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A UNIVERSAL SEQUENCE OF PERIOD-DOUBLING BIFURCATIONS OF THE FORCED OSCILLATIONS OF A PENDULUM*

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One of the most typical modes of chaotization in deterministic systems occurs when variation of the parameter characterizing the intensity of a disturbance takes a dynamical system through a sequence of period-doubling bifurcations from a regular to a stochastic mode of behaviour. The transition occurs in regions of phase space characterized by a strong local instability and obeys the law of universality recently discovered by Feigenbaum /1/.

In this paper continuation with respect to a parameter and methods of branching theory /3/ are used in combination to construct a sequence of period-doubling bifurcations for the forced oscillations of a conservative pendulum. This sequence is shown to possess the universality property.

1. We shall be concerned in this paper with the evolution and bifurcation of periodic solutions of the equation of forced oscillations of a pendulum

$$x^{\cdot\cdot} + k^2 \sin x = \lambda \sin \omega t \tag{1.1}$$

when the parameter λ is varied and with analysis of the stability of these solutions. Suppose that at some parameter value $\lambda = \lambda_{(0)}$ the pendulum oscillates with period $T = 2\pi/\omega$. The corresponding T-periodic solution ("T-solution") of Eq.(1.1) satisfies the conditions

$$x_{(0)}(0) = x_{(0)}(T), \quad x_{(0)}^{\cdot}(0) = x_{(0)}^{\cdot}(T) \tag{1.2}$$

The solutions of Eq.(1.1) are continuous functions of the initial conditions and the parameter λ, so that the T-periodicity conditions can be written

$$\begin{gathered} x_0 = x\ (x_0, x_0^{\cdot}, \lambda, T) \\ x_0^{\cdot} = x^{\cdot}(x_0, x_0^{\cdot}, \lambda, T) \quad (x_0 = x(0)) \end{gathered} \tag{1.3}$$

We now vary both sides of (1.3) in the neighbourhood of the state $\lambda = \lambda_{(0)}$, $x_0 = x_{(0)0}$, $x_0^{\cdot} = x_{(0)0}^{\cdot}$

$$\begin{gathered} \delta x_{(0)0} = \frac{\partial x(T)}{\partial x_0}\delta x_{(0)0} + \frac{\partial x(T)}{\partial x_0^{\cdot}}\delta x_{(0)0}^{\cdot} + \frac{\partial x(T)}{\partial \lambda}\delta\lambda_{(0)} \\ \delta x_{(0)0}^{\cdot} = \frac{\partial x^{\cdot}(T)}{\partial x_0}\delta x_{(0)0} + \frac{\partial x^{\cdot}(T)}{\partial x_0^{\cdot}}\delta x_{(0)0}^{\cdot} + \frac{\partial x^{\cdot}(T)}{\partial \lambda}\delta\lambda_{(0)} \\ (x_{(0)0} = x_{(0)}(0)) \end{gathered} \tag{1.4}$$

Introducing the notation $\partial x(t)/\partial x_0 = y_1(t)$, $\partial x(t)/\partial x_0^{\cdot} = y_2(t)$, $\partial x(t)/\partial\lambda = y_\lambda(t)$, we determine $y_1(T)$, $y_2(T)$, $y_\lambda(T)$ from the solutions of the appropriate variational equations

$$\begin{gathered} y_1^{\cdot\cdot} + k^2 y_1 \cos x = 0, \quad y_{10} = 1, \quad y_{10}^{\cdot} = 0 \\ y_2^{\cdot\cdot} + k^2 y_2 \cos x = 0, \quad y_{20} = 0, \quad y_{20}^{\cdot} = 1 \end{gathered} \tag{1.5}$$

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$$y_\lambda^{\cdot\cdot} + k^2 y_\lambda \cos x = \sin \omega t, \quad y_{\lambda 0} = 0, \quad y_{\lambda 0}^{\cdot} = 0 \tag{1.6}$$

Specifying the value of $\delta\lambda_{(0)}$, we obtain the variations of the initial conditions $\delta x_{(0)0}$, $\delta x_{(0)0}^{\cdot}$ as a solution of the system of linear equations

$$\begin{gathered}[y_1(T) - 1]\,\delta x_{(0)0} + y_2(T)\,\delta x_{(0)0}^{\cdot} = -y_\lambda(T)\,\delta\lambda_{(0)} \\ y_1^{\cdot}(T)\,\delta x_{(0)0} + [y_2^{\cdot}(T) - 1]\,\delta x_{(0)0}^{\cdot} = -y_\lambda^{\cdot}(T)\,\delta\lambda_{(0)}\end{gathered} \tag{1.7}$$

The solution of the Cauchy problem for Eq.(1.1) with initial conditions $x_{(1)0} = x_{(0)0} + \delta x_{(0)0}$, $x_{(1)0}^{\cdot} = x_{(0)0}^{\cdot} + \delta x_{(0)0}^{\cdot}$ with $\lambda_{(1)} = \lambda_{(0)} + \delta\lambda_{(0)}$ will satisfy the periodicity conditions (1.3) with a (residual) error

$$x_{(1)0} = x_{(1)}(T) + r_{(0)}^1, \quad x_{(1)0}^{\cdot} = x_{(1)}^{\cdot}(T) + r_{(0)}^2$$

of the same order as the terms of the Taylor series omitted in (1.4). The error in the initial conditions is adjusted by use of the Newton-Kantorovich method, and then (k is the number of iterations)

$$x_{(1)0} = x_{(0)0} + \delta x_{(0)0} + \sum_{j=0}^{k-1} \delta x_{r(j)0}$$

$$x_{(1)0}^{\cdot} = x_{(0)0}^{\cdot} + \delta x_{(0)0}^{\cdot} + \sum_{j=0}^{k-1} \delta x_{r(j)0}^{\cdot}$$

$$\begin{Vmatrix}\delta x_{r(j)0} \\ \delta x_{r(j)0}^{\cdot}\end{Vmatrix} = -\begin{Vmatrix} y_1(T) - 1 & y_2(T) \\ y_1^{\cdot}(T) & y_2^{\cdot}(T) - 1\end{Vmatrix}^{-1} \begin{Vmatrix} r_{(j)}^{(1)} \\ r_{(j)}^{2}\end{Vmatrix}$$

The process is now continued, using $x_{(1)0}$, $x_{(1)0}^{\cdot}$ as a generating solution, putting $\lambda = \lambda_{(1)}$ and subjecting λ to a further variation, to find T-solutions for $\lambda = \lambda_{(2)} = \lambda_{(1)} + \delta\lambda_{(1)}$; and so on.

The stability of these periodic solutions, and conditions for their existence and uniqueness in the relevant neighbourhood, may be investigated relying on Floquet's Theorem, by analysing the multipliers ρ_i of the monodromy matrix /4/

$$Y(T) = \begin{Vmatrix} y_1(T) & y_2(T) \\ y_1^{\cdot}(T) & y_2^{\cdot}(T)\end{Vmatrix} \tag{1.8}$$

A period-doubling bifurcation occurs in a conservative second-order system in states for which both multipliers of the matrix (1.8) are equal to $\rho_{1,2} = -1$. Once this state is reached, the construction goes on with period $2T$. In this state

$$\det \| Y(2T) - E \| = 0$$

and therefore higher-order terms must be retained when using the representation (1.4) ($y_1, y_2, \ldots, y_{2\lambda}$ are defined at $t = 2T$):

$$\begin{gathered}\delta x_0 = y_1 \delta x_0 + y_2 \delta x_0^{\cdot} + y_\lambda \delta\lambda + \\ {}^1/_2 y_{11} (\delta x_0)^2 + {}^1/_2 y_{22} (\delta x_0^{\cdot})^2 + {}^1/_2 y_{\lambda\lambda} (\delta\lambda)^2 + \\ y_{12} \delta x_0 \delta x_0^{\cdot} + y_{1\lambda} \delta x_0 \delta\lambda + y_{2\lambda} \delta x_0^{\cdot} \delta\lambda + \ldots \\ \delta x_0^{\cdot} = y_1^{\cdot} \delta x_0 + y_2^{\cdot} \delta x_0^{\cdot} + y_\lambda^{\cdot} \delta\lambda + \\ {}^1/_2 y_{11}^{\cdot} (\delta x_0)^2 + {}^1/_2 y_{22}^{\cdot} (\delta x_0^{\cdot})^2 + {}^1/_2 y_{\lambda\lambda}^{\cdot} (\delta\lambda)^2 + \\ y_{12}^{\cdot} \delta x_0 \delta x_0^{\cdot} + y_{1\lambda}^{\cdot} \delta x_0 \delta\lambda + y_{2\lambda}^{\cdot} \delta x_0^{\cdot} \delta\lambda + \ldots\end{gathered} \tag{1.9}$$

The coefficients of the higher-order terms in Eqs.(1.9) are the derivatives of the functions y_1, y_2, y_λ with respect to $x_0, x_0^{\cdot}$ and λ at $t = 2T$. A subscript 1 indicates differentiation with respect to x_0, a subscript 2 with respect to $x_0^{\cdot}$, and subscript λ with respect to λ. To determine these coefficients we differentiate the variational Eqs.(1.5), (1.6) term by term with respect to the initial conditions and the parameter, obtaining the following system of equations:

$$\begin{gathered}y_{\xi\xi}^{\cdot\cdot} + k^2 y_{\xi\xi} \cos x = k^2 y_\xi^2 \sin x, \quad y_{\xi\xi 0} = 0, \quad y_{\xi\xi 0}^{\cdot} = 0 \\ y_{\eta\zeta}^{\cdot\cdot} + k^2 y_{\eta\zeta} \cos x = k^2 y_\eta y_\zeta \sin x, \quad y_{\eta\zeta 0} = 0, \quad y_{\eta\zeta 0}^{\cdot} = 0 \\ (\xi = 1, 2, \lambda, \ \eta = 1, 2, \ \zeta = 2, \lambda; \ \eta \neq \zeta)\end{gathered} \tag{1.10}$$

The system of branching Eqs.(1.8) /3/ with coefficients $y_1(2T), y_2(2T), \ldots, y_{2\lambda}(2T)$ calculated from (1.5), (1.6) and (1.10) and prescribed $\delta\lambda$ is solved by separation of roots with subsequent Newton-Kantorovich correction. After that, the solution can be extended along each branch by the continuation with respect to the parameter. If system (1.9) has no solutions or if there are multiple roots, further terms of the expansion must be included in the branching equations and the computations repeated.

Extending a $2T$-solution of Eq.(1.1) as a function of λ, we find a bifurcation value of λ corresponding to the next period doubling of the forced oscillations; and so on.

This method of constructing a sequence of period-doubling bifurcations for oscillations of a mechanical system, followed by stability analysis of the 2^nT-solutions of Eq.(1.1) thus obtained, is specially designed for high-speed computer application and high-precision computing procedures.

A few words about the main features of numerical implementation of our method. An ALGOL program was written to construct a bifurcation "tree" of periodic solutions of Eq.(1.1) as λ is varied. All computations are done with double precision. Eqs.(1.1), (1.5) and (1.6), and at branch points also Eqs.(1.9), were integrated jointly by the Everhart method /5/ with high precision (~12-13 accurate digits at the end of a period). The continuation step $\delta\lambda_{(m)}$ was chosen so that after two or three Newton-Kantorovich iterations the following conditions were satisfied:

$$|r^{(1)}_{(k)}| \leqslant \mu \max_t |x_{(k)}(t)|, \quad |r^{(2)}_k| \leqslant \mu \max_t |\dot{x}_{(k)}(t)| \quad (\mu \approx 10^{-10})$$

In the neighbourhood of the bifurcation values of λ the accuracy of the solution was improved by the dichotomy method.

2. Our construction of a sequence of period-doubling bifurcations for the forced oscillations of a pendulum was tried out at $k^2 = 1$, $\omega = 1.3$. As generating T-solution $(T = 2\pi/\omega)$ we took the stable equilibrium state $\lambda_{(0)} = 0$, $x_{(0)0} = 0$, $\dot{x}_{(0)0} = 0$.

An increase in the intensity of the applied disturbance $\lambda > \lambda_{(0)}$ induced an increase in increase in amplitude and a decrease in the frequency $k^* < k$ of free non-linear oscillations, since the amplitude-frequency characteristic of the pendulum is smooth. Thus, if the frequency of the applied disturbance is taken to be $\omega > k$, the system retreats from non-linear resonance as λ increases.

We will now trace the stability of T-periodic oscillations as λ increases. On the branch of T-periodic oscillations corresponding to the interval $\lambda_{(0)} \leqslant \lambda < 1.725...$ complex-conjugate multipliers lie on the unit circle. Such oscillations are therefore stable. They have a phase portrait of the type shown in Fig.1a.

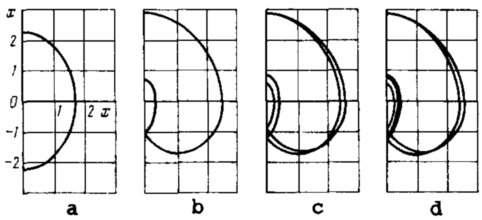

Fig.1

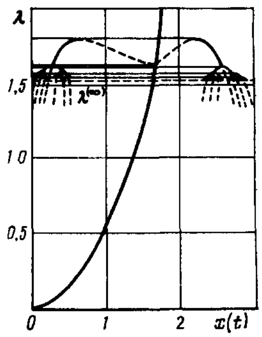

Fig.2

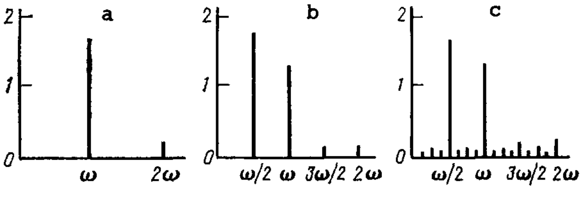

Fig.3

At $\lambda = \lambda^{(1)}$ (see the table) the multipliers assumed a value $\rho_1 = \rho_2 = -1$, indicating that bifurcation of the solution is accompanied by period doubling. The branch of $2T$-solutions branching from this state turned out to be unstable, since one of the multipliers computed at period $2T$ lay outside the unit circle.

i	$\lambda^{(i)}$	$\delta^{(i)}$	$x_0^{(i)}$	$-x_0^{\cdot(i)}$
1	1,725	———	0	2,26561727272
1′	1.77095912802	7,12337865	1,31201	1.86039918519
2	1,72462398665	8,21381505	1,73337025268	1.53906612857
3	1,718119328404	8,70705458	1,82946173268	1.48415882553
4	1,7173274115897	8,71389854	1,83418944574	1.48435587447
5	1,71723646041822	8,72092405	1,83339655541	1 48493217056
6	1,71722602293480	8,72109726	1,83354848610	1,48490249131
7	1,71722482610252		1,83359211395	1.48488615372
8	1.71722468886838		1,83359422288	1,48488506144

We nevertheless traced the further evolution of the system as the amplitude of the applied disturbance was varied. The branch of $2T$-solutions remains unstable until λ becomes equal to $\lambda^{(1')}$. At $\lambda = \lambda^{(1')}$ there is a limit point on this branch, at which $\rho_1 = \rho_2 = 1$. Continuation of the branch of $2T$-solutions beyond the limit point is accompanied by a decrease in λ to the value $\lambda^{(2)}$; in the interval $\lambda^{(1')} > \lambda > \lambda^{(2)}$ the corresponding $2T$-periodic oscillations of the pendulum are stable. At $\lambda = \lambda^{(2)}$ the stable $2T$-solution, whose phase portrait is shown in Fig.1b, undergoes a period doubling bifurcation, generating a stable $4T$-solution. This branch of $4T$-solutions is stable in the interval $\lambda^{(2)} > \lambda > \lambda^{(3)}$. The parameter value $\lambda^{(3)}$ corresponds to the next period doubling bifurcation and the appearance of a stable $8T$-solution; and so on.

The eight first period-doubling bifurcations were found for this particular conservative pendulum. The bifurcation values of the parameter $\lambda^{(i)}$ $(i = 1, 2, \ldots, 8)$ are listed in the table. Also listed are the initial conditions of the $2^{(i-1)}T$-solutions at the bifurcation points. Note that in all intervals $\lambda^{(i-1)} > \lambda > \lambda^{(i)}$ (except in the case $i = 2$) the corresponding $2^{(i-1)}T$-periodic oscillations of the pendulum are stable.

The phase trajectories of T- to $8T$-solutions in the right half-plane corresponding to bifurcation values $\lambda^{(i)}$ $(i = 1, 2, 3, 4)$ are shown in Fig.1a-d. Fig.2 illustrates the branching tree in the right half-plane for Eq.(1.1), showing a section of the space $x(t), x^{\cdot}(t), \lambda$ by the plane $x^{\cdot}(t) = 0$. The phase portraits and branching tree are symmetrical about the axis $0x$. Branches corresponding to stable $2^{(i-1)}T$-solutions are indicated by the solid curves, and unstable ones by the dashed curves. It can be seen from the phase portraits of the computed $2^{(i-1)}T$-solutions that each further period-doubling bifurcation splits the phase trajectory of the system and reduces the regularity of the motion. However, as i increases the distances between adjacent sections of the phase trajectories decrease rapidly and by $i > 5$ they are hardly distinguishable. In this connection we note that the phase trajectory of the $128T$-solution cuts the x axis at 256 points.

Analysis of the bifurcation values $\lambda^{(i)}$ shows that the sequence converges to a certain point of accumulation $\lambda^{(\infty)}$ /6/ according to a law close to that of a geometric progression with ratio

$$\delta^{(i)} = (\lambda^{(i)} - \lambda^{(i+1)})/(\lambda^{(i+1)} - \lambda^{(i+2)})$$

and the exponent $\delta^{(i)}$ itself converges as i increases to Feigenbaum's universal conservative constant $\delta = 8.7210972\ldots$, /7/, obtained for two-dimensional Hamiltonian maps. It is typical that as early as $i = 6$ $\delta^{(i)}$ is identical with δ up to the eighth significant digit (table).

The high rate of convergence δ of the bifurcation values λ to the accumulation point $\lambda^{(\infty)}$ implies that even slight variations in the intensity of the applied disturbance cause a rapid decrease in the regularity of the motion and produce highly involved phase trajectories. A glance at Fig.2 reveals the cascade-like nature of the process.

Using the constant δ /7/, one can determine the approximate value of $\lambda^{(\infty)}$ at which the oscillations of the pendulum become chaotic. Thus, at $i = 7$,

$$\lambda^{(\infty)} \approx \lambda^{(i)} - (\lambda^{(i)} - \lambda^{(i+1)})\, \delta/(\delta - 1) = 1.71722467109446$$

Harmonic analysis of the computed periodic solutions produces the Fourier spectra shown in Fig.3a-c for the $2^{(i-1)}T$-solutions of Eq.(1.1) at $i = 1, 2, 4$, respectively. A charcteristic feature of the spectra is the appearance of subharmonics at frequencies $\omega/2^i$ and multiples of their odd harmonics at the i-th period doubling bifurcation. As λ is increased, the amplitude of each harmonic increases up to a certain saturation level, which in turn decreases as i increases. As $i \to \infty$ the spectrum becomes continuous, indicating the chaotic nature of the oscillations.

Hence, our study has confirmed the possibility that the transition to chaos in the oscillations of a pendulum occurs by way of a Feigenbaum-universal sequence of period-doubling bifurcations.

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LIMIT CYCLES AND CHAOS IN EQUATIONS OF THE PENDULUM TYPE*

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It is proved that for sufficiently small ε the equation

$$x^{\cdot\cdot} + \sin x = \varepsilon x^{\cdot} \cos nx, \quad n \in N \tag{0.1}$$

where ε is a parameter, has exactly $n-1$ coarse limit cycles (l.c.'s) in the region of oscillatory motions and no l.c.'s in the region of rotary motions (i.e., l.c.'s going round the phase cylinder). This result is used to study an equation of type (0.1) with time-periodic term on the right. The role of l.c.'s in the formation of quasi-attractors (q.a.'s) is demonstrated. A computer-generated description is given of the process by which q.a.'s with developed chaos are formed (for $n=3$).

1. Statement of the problem. Main results. We consider equations of the form

$$x^{\cdot\cdot} + A(x) = \varepsilon f(x, x^{\cdot}, \nu t; \varepsilon) \tag{1.1}$$

where A is a 2π-periodic function of x and f a periodic function of x and $\varphi = \nu t$ with the same period; ε, ν are parameters. Equations of this kind govern the motions of various pendulums. Among other applications we mention the problem of the structure of resonance zones in non-conservative time-periodic systems

$$\frac{du}{d\tau} = \frac{\partial H(u,v)}{\partial v} + \mu R(u, v, \tau), \quad \frac{dv}{d\tau} = -\frac{\partial H(u,v)}{\partial u} + \mu G(u, v, \tau) \tag{1.2}$$

where μ is a small parameter. As shown in /1, 2/, this problem involves investigating an equation of the form (1.1) with a small parameter ε depending on μ. In addition, $f = \sigma(x) x^{\cdot} + O(\varepsilon)$, where $\sigma(x)$ is defined by the divergence of the vector field of system (1.2).

We set $A(x) = \sin x$ and consider, first of all, the case in which ε is a small parameter Eq.(1.1) has been studied in this case /3/ for a special form of the function f. A more general setting was considered in /2/. It has been observed that an important role in the study of Eq.(1.1) is played by the l.c.'s of the autonomous equation

$$x^{\cdot\cdot} + \sin x = \varepsilon f_0(x, x^{\cdot}), \quad f_0 = \langle f(x, x^{\cdot}, \varphi; 0)\rangle_{\varphi} \tag{1.3}$$

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